**Mathematics 433** **Fall 2006** **Problems to Turn in: 10**

**November 17, 2006**

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**Name**

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"*Personally, I'm always ready to learn, although I do not always like being taught.*" – Winston Churchill

## Problems

1. Let $S$ be any set and denotethe group of permutations on $S$ by $Perm(S)$. Prove that there is a one-to-one and onto function $\psi : S \to Perm(S)$ given by the rule $\psi(g) = m_g$. Here $m_g(s) = gs$ for all $s \in S$.

2. Prove that a homomorphism $\psi : G \to Perm(S)$ (where $S$ is a fixed set and $G$ is a group) is injective if and only if the group action of $G$ on $S$ satisfies the following property: If $gs = s$ for every $s \in G$, then $g = e$.

3. Prove that the group $GL(2, \mathbf{F}_2)$ of invertible matrices with mod 2 coefficients is isomorphic to the symmetric group $S^3$.

4. Let $G$ be the group of rotational symmetries of a cube $C$. Two regular tetrahedra $\Delta$ and $\Delta'$ can be inscribed in $C$, each using half of the vertices. What is the order of the stabilizer of $\Delta$?

5. Do **one** of the following.

   (a) Prove the formula $|G| = |Z(G)| + \sum |C|$ where the sum is over the conjugacy classes containing more than one element and $Z(G)$ is the center of $G$.

   (b) Rule out as many of the following as possible as Class Equations for a group of order 10.

      i. $1+1+1+2+5$

      ii. $1+2+2+5$

      iii. $1+2+3+4$

      iv. $1+1+2+2+2+2$

6. Let $Z(G)$ be the center of a group $G$. Prove that if $G/Z$ is a cyclic group, then $G$ is abelian and hence $G = Z(G)$.

## Problems from Turn In Set 09 – Many that weren't used.

1. (A useful result for later) Suppose $p$ is a prime integer, $G$ is a group, $x \in G$ is an element of order $p$ and $y \in G$ is an element of order $p$ but $y \notin \langle x \mid x^p = e\rangle$. Prove or disprove: $\langle x \mid x^p = e\rangle \cap \langle y \mid y^p = e\rangle = \{e\}$.

2. (This problem generalizes our proof that the center of a $p$ - group has order greater than 1.) Let $G$ be a $p$ - group and let $S$ be a finite set on which $G$ acts. Assume the order of $S$ is **not** divisible by $p$.Then there is a fixed point for the action of $G$ on $S$. That is, there is an element $s \in S$ whose stabilizer is the whole group.

3. Prove:

(a) No group of order $p^2q$, where $p$ and $q$ are prime, is simple.

(b) No group of order 224 is simple.

4. Let $G$ be a group of order $p^l m$. Our textbook (Gallian) contains an argument that $G$ contains a subgroup of order $p^r$ for every integer $1 \leq r \leq l$. Finish this argument by proving exercise 45 in chapter 10 of Gallian. That is, Let $N$ be a normal subgroup of a group $G$. Use property 7 of Theorem 10.2 to prove every subgroup of $G/N$ has the form $H/N$ where $H$ is a subgroup of $G$.

5. Do a. , b. , or c. of the following.

   (a) Let $H_1, \cdots, H_k$ be a complete list of all $p$ - Sylow subgroups of a finite group $G$. Prove $H = \cap_{i=1}^{k} H_i$ is a normal subgroup of $G$.

   (b) Prove the only simple groups of order less than 60 are groups of prime order.

   (c) Classify all groups of order 18.

6. Do any of the three choices in problem 5 that you didn't do for problem 5.

7. Let $G$ be the group of rotational symmetries of a cube $C$. Two regular tetrahedra $\Delta$ and $\Delta'$ can be inscribed in $C$, each using half of the vertices. What is the order of the stabilizer of $\Delta$?

8. Rule out as many of the following as possible as Class Equations for a group of order 10.

   (a) i. $1 + 1 + 1 + 2 + 5$
   ii. $1 + 2 + 2 + 5$
   iii. $1 + 2 + 3 + 4$
   iv. $1 + 1 + 2 + 2 + 2 + 2$

9. (This is a theorem in Gallian: don't assign as a problem).

   Let $Z(G)$ be the center of a group $G$. Prove that if $G/Z$ is a cyclic group, then $G$ is abelian and hence $G = Z(G)$.

10. Let $X$ be a path-connected topological space, $x_0$ a fixed point of $X$ from which all loops start and stop and $\pi(X.x_0)$ the equivalence classes of loops outlined in class for the fundamental group. Use a homotopy to show the products of equivalence classes of loops are associative. That is, show $(f * g) * h$ is homotopic to $f * (g * h)$.